Gauss-Zeidel method for solving linear systems of equations

The Gauss-Zeidel method is a modification of the iteration methods that are the same type as the method of consecutive approximations and the method of the simple iteration (Jacobi).

Gauss-Zeidel scheme for the method of consecutive approximations:

$$x_i^{(k)} = \sum_{j=1}^{i-1} b_{ij} x_j^{(k)} + \sum_{j=i}^{n} b_{ij} x_j^{(k-1)} + b_i\,, \qquad i = 1, 2, \ldots, n; \qquad k = 1, 2, \ldots$$

or in an expanded form

$$\left|\begin{array}{llllllll}
x_1^{(k)} = & b_{11}x_1^{(k-1)} & + b_{12}x_2^{(k-1)} & + b_{13}x_3^{(k-1)} + & \ldots & + b_{1n-1}x_{n-1}^{(k-1)} & + b_{1n}x_n^{(k-1)} & + b_1 \\
x_2^{(k)} = & b_{21}x_1^{(k)} & + b_{22}x_2^{(k-1)} & + b_{23}x_3^{(k-1)} + & \ldots & + b_{2n-1}x_{n-1}^{(k-1)} & + b_{2n}x_n^{(k-1)} & + b_2 \\
x_3^{(k)} = & b_{31}x_1^{(k)} & + b_{32}x_2^{(k)} & + b_{33}x_3^{(k-1)} + & \ldots & + b_{3n-1}x_{n-1}^{(k-1)} & + b_{3n}x_n^{(k-1)} & + b_3\,, \\
 & \ldots\ldots & & & & & & \\
x_n^{(k)} = & b_{n1}x_1^{(k)} & + b_{n2}x_2^{(k)} & + b_{n3}x_3^{(k)} + & \ldots & + b_{nn-1}x_{n-1}^{(k)} & + b_{nn}x_n^{(k-1)} & + b_n
\end{array}\right.$$

$$k = 1, 2, \ldots$$

Gauss-Zeidel scheme for the method of simple iteration:

$$x_i^{(k)} = \sum_{j=1}^{i-1} c_{ij} x_j^{(k)} + \sum_{j=i}^{n} c_{ij} x_j^{(k-1)} + d_i\,, \qquad i = 1, 2, \ldots, n; \qquad k = 1, 2, \ldots$$

In this case the expanded form is the same as the previous one. In both methods the Gauss-Zeidel modification is subject to the following scheme for finding a k^{th} approximation:

$$\begin{array}{ccccccccccc}
x_1^{(k-1)} & , & x_2^{(k-1)} & , & x_3^{(k-1)} & ,\ldots, & x_{n-1}^{(k-1)} & , & x_n^{(k-1)} & \to & x_1^{(k)} \\
\boxed{x_1^{(k)}} & , & x_2^{(k-1)} & , & x_3^{(k-1)} & ,\ldots, & x_{n-1}^{(k-1)} & , & x_n^{(k-1)} & \to & x_2^{(k)} \\
\boxed{x_1^{(k)}} & , & \boxed{x_2^{(k)}} & , & x_3^{(k-1)} & ,\ldots, & x_{n-1}^{(k-1)} & , & x_n^{(k-1)} & \to & x_3^{(k)} \\
\ldots\ldots\ldots & & & & & & & & & &
\end{array}$$

$$\boxed{x_1^{(k)}}\quad,\quad\boxed{x_2^{(k)}}\quad,\quad\boxed{x_3^{(k)}}\quad,\ldots,\quad\boxed{x_{n-1}^{(k)}}\quad,\quad x_n^{(k-1)}\quad\rightarrow\quad x_n^{(k)}$$

The conditions for convergence of the method of consecutive approximations and the Jacobi method are **sufficient conditions** for the convergence of the Zeidel modification. In most cases the Zeidel modification is faster, but not in order:

> The Gauss-Zeidel modification can converge when the Jacobi method and the method of consecutive approximations are not.

Example. For the given system apply the Gauss-Zeidel modification for the Jacobi method. Work with an accuracy of six digits after the decimal point, using the zero vector as initial guess. Make six iterations.

$$\left|\begin{array}{llll} 4x_1 & -x_2 & & =2 \\ -x_1 & +4x_2 & -x_3 & =6 \\ & -x_2 & +4x_3 & =2 \end{array}\right. .$$

Solution:

$\boxed{1.}$ $A=\begin{pmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{pmatrix} \rightarrow C=\begin{pmatrix} 0 & 0,25 & 0 \\ 0,25 & 0 & 0,25 \\ 0 & 0,25 & 0 \end{pmatrix};\ b=\begin{pmatrix} 2 \\ 6 \\ 2 \end{pmatrix} \rightarrow d=\begin{pmatrix} 0,5 \\ 1,5 \\ 0,5 \end{pmatrix}.$

$\boxed{2.}$ Convergence check – done (see the same example in the theme Jacobi Method).

$\boxed{3.}$ Execution of six iterations:

$$\left|\begin{array}{lllll} x_1^{(1)}= & & 0,25x_2^{(0)} & & +0,5 =0,25.0 \quad +0,5=0,5 \\ x_2^{(1)}= & 0,25x_1^{(1)} & & +0,25x_3^{(0)} & +1,5 =0,25.0,5+0,25.0+1,5=1,625 \\ x_3^{(1)}= & & 0,25x_2^{(1)} & & +0,5 =0,25.1,625+0,5=0,90625 \end{array}\right. \rightarrow \boxed{\begin{array}{l} x_1^{(1)}=0,5 \\ x_2^{(1)}=1,625 \\ x_3^{(1)}=0,90625 \end{array}}.$$

$$\left|\begin{array}{lllll} x_1^{(2)}= & & 0,25x_2^{(1)} & & +0,5= \quad 0,25.(1,625)\ +0,5=0,90625 \\ x_2^{(2)}= & 0,25x_1^{(2)} & & +0,25x_3^{(1)} & +1,5=0,25.(0,90625)+0,25.(0,90625)\ +1,5=1,953125 \\ x_3^{(2)}= & & 0,25x_2^{(2)} & & +0,5= \quad 0,25.(1,953125)+0,5=0,988281 \end{array}\right.$$

$$\rightarrow \boxed{\begin{aligned} x_1^{(2)} &= 0{,}90625 \\ x_2^{(2)} &= 1{,}953125 \\ x_3^{(2)} &= 0{,}988281 \end{aligned}}\ .$$

Do the other iterations independently. The results are printed in the following table. Do one more thing: compare these results with the results of table 2 in the theme Jacobi method and make your conclusion.

Table 3

k \ $x^{(k)}$	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$
0	0	0	0
1	0,5	1,625	0,90625
2	0,90625	1,953125	0,988281
3	0,988281	1,994141	0,998535
4	0,998535	1,999268	0,999817
5	0,999817	1,999908	0,999977
6	0,999977	1,999989	0,999997
...	...	...	...
x^*	1	2	1

Author: Iliya Makrelov, ilmak@uni-plovdiv.bg